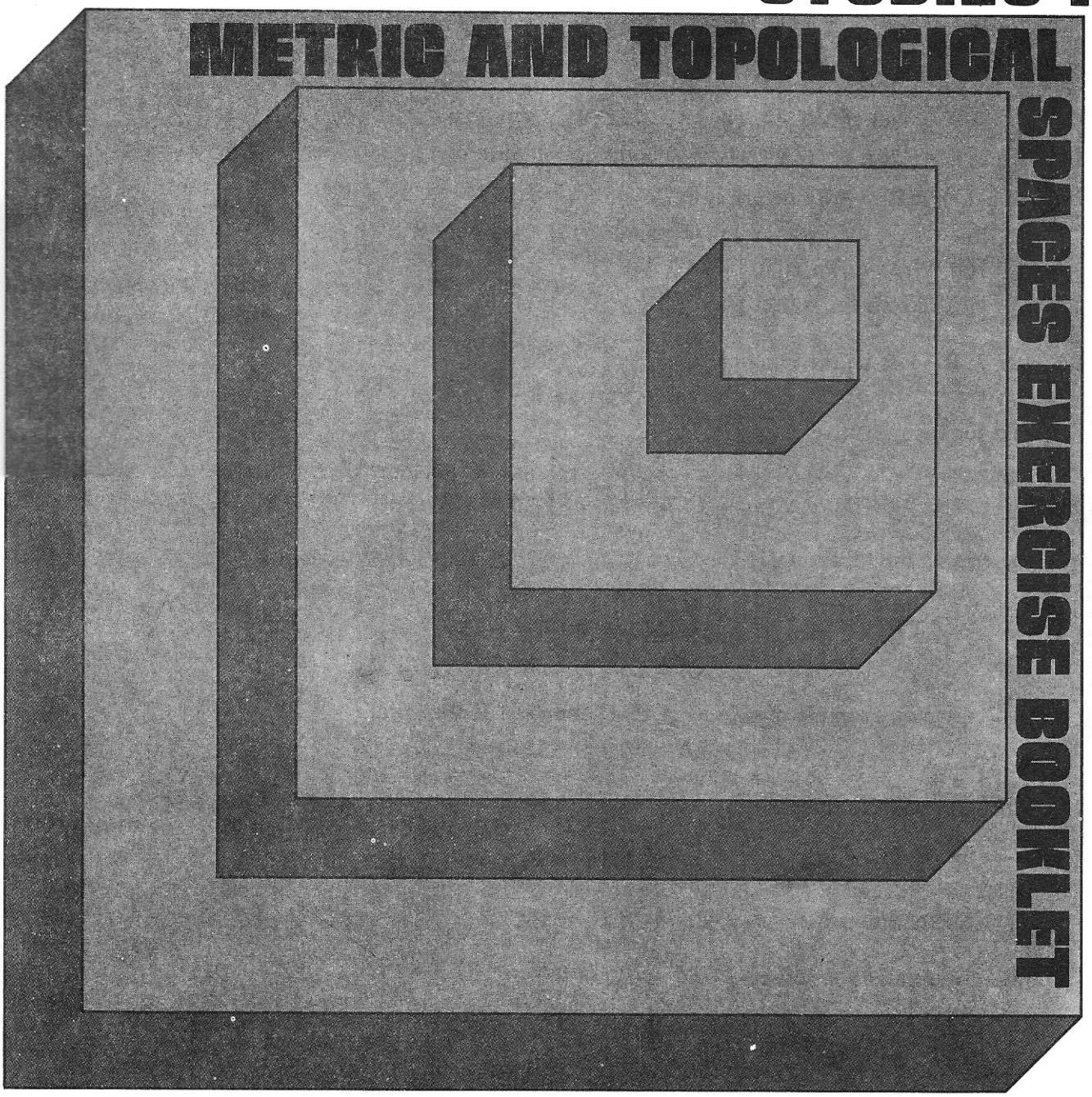
M335 MTS EB

THE OPEN UNIVERSITY
Mathematics: A Third Level Course
M335 Studies in Pure Mathematics

STUDIES IN PURE MATHEMATICS

METRIC AND TOPOLOGICAL SPACES EXERCISE BOOKLET



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1.2

Unit 1

1.1 (i) Let $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ be two indexed families of sets with the same indexing set I. Prove that

$$\bigcup_{i\in I}(A_i - B_i) \subset \left(\bigcup_{i\in I} A_i\right) - \left(\bigcap_{i\in I} B_i\right).$$

(ii) Give an example of subsets A_1, A_2, B_1, B_2 of $\{1, 2, 3, 4\}$ such that

$$(A_1 - B_1)\cup(A_2 - B_2) \neq (A_1 \cup A_2) - (B_1 \cap B_2).$$

1.2 For each positive integer n, let

$$A_n = \{(x_1, x_2)\in\mathbf{R}^2 : x_2 \geq 0, nx_1 + x_2 \geq 0\}.$$

(i) Sketch the subsets A_1, A_2 of $\mathbf{R}^2$.

(ii) Show that

$$\bigcap_{n=1}^{\infty} A_n = \{(x_1, x_2)\in\mathbf{R}^2 : x_1 \geq 0, x_2 \geq 0\}.$$

1.3 (i) Let $f: \mathbf{R}^2 \longrightarrow \mathbf{R}$ be given by

$$f(x_1, x_2) = x_1^2 + x_2^2, \quad (x_1, x_2)\in\mathbf{R}^2.$$

Find: (a) $f^{-1}([1, 4])$;
(b) $f^{-1}(\{-4\})$;
(c) $f^{-1}([-4, 1))$;
(d) $f^{-1}(\mathbf{Z})$, where $\mathbf{Z}$ is the set of all integers.

(ii) Let $g: \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ be given by

$$g(x_1, x_2) = (x_1^2 + x_2^2, x_1x_2), \quad (x_1, x_2)\in\mathbf{R}^2.$$

(a) Write down $g([0, 1] \times [0, 1])$.
(b) Sketch the subset $g^{-1}([1, 4] \times [0, 1])$ of $\mathbf{R}^2$.

1.4 Let $f: \mathbf{R} \longrightarrow \mathbf{R}$ be the function given by

$$f(x) = 1 - x^2, \quad x\in\mathbf{R},$$

and for each positive integer n, let

$$A_n = \left[0, 1 - \frac{1}{n^2}\right].$$

Determine the subset $\bigcup_{n=1}^{\infty} f^{-1}(A_n)$ of $\mathbf{R}$.

1.5 (i) Prove that if A, B, C and D are sets, then

$$(A \times C)\cup(B \times D) \subset (A \cup B) \times (C \cup D).$$

(ii) Give an example of subsets A, B, C and D of $\mathbf{R}$ such that

$$(A \times C)\cup(B \times D) \neq (A \cup B) \times (C \cup D).$$

1.6 For each positive integer n, let

$$A_n = \left[-\frac{1}{n}, \frac{1}{n}\right], \quad B_n = \left[-\frac{n}{n+1}, \frac{n}{n+1}\right].$$

(i) Sketch the subsets $A_1 \times B_1$, $A_2 \times B_2$ of $\mathbf{R}^2$.

(ii) Determine the subset $\bigcap_{n=1}^{\infty}(A_n \times B_n)$ of $\mathbf{R}^2$.

Unit 2

2.1 Let A and B be sets of real numbers which are bounded above, and suppose that $A \cap B$ is non-empty.

(i) Prove that the sets $A \cap B$ and $A \cup B$ are bounded above.

(ii) Prove that

$$\sup(A \cup B) = \text{Max}\{\sup A, \sup B\}.$$

(iii) Prove that

$$\sup(A \cap B) \leq \text{Min}\{\sup A, \sup B\}.$$

(iv) Find sets A and B for which equality does not hold in part (iii).

2.2 (i) Let A be a non-empty set of real numbers which is bounded below by α. Prove that $\inf A = \alpha$ if and only if for each $\varepsilon > 0$, we can find $a \in A$ such that

$$\alpha \leq a < \alpha + \varepsilon.$$

(ii) Let A, B be non-empty sets of real numbers which are bounded below, and let C be the set of all numbers of the form $3a + 2b$ where $a \in A$, $b \in B$.

(a) Show that C is bounded below.

(b) Prove that if $\alpha = \inf A$ and $\beta = \inf B$, then $\inf C = 3\alpha + 2\beta$.

2.3 For $n = 1, 2, 3, \ldots$, let

$$s_n = \frac{2n - 1}{3n + 1}.$$

(i) Prove that (s_n) is a monotonic increasing sequence.

(ii) Show that for every positive integer n,

$$\frac{2}{3} - \frac{1}{n} < s_n < \frac{2}{3}.$$

(iii) Prove that (s_n) converges to $\frac{2}{3}$ by using an ε-N argument.

2.4 (i) Let f: $\mathbf{R} \longrightarrow \mathbf{R}$ be given by

$$f(x) = 1 + x^2, \quad x \in \mathbf{R}.$$

Show that given $\varepsilon > 0$, $|f(x) - f(1)| < \varepsilon$ whenever $|x - 1| < \delta$, if we take $\delta = \text{Min}\{1, \varepsilon/3\}$.

(ii) For g: $\mathbf{R} \longrightarrow \mathbf{R}$ defined by

$$g(x) = 1 + x^3, \quad x \in \mathbf{R},$$

given $\varepsilon > 0$, find a $\delta > 0$ such that $|g(x) - g(1)| < \varepsilon$ whenever $|x - 1| < \delta$.

(iii) Deduce that each of the functions h_1, h_2, h_3: $\mathbf{R} \longrightarrow \mathbf{R}$, given by

$$h_1(x) = 2 + x^2 + x^3,$$
$$h_2(x) = 1 + x^2 + x^3 + x^5,$$
$$h_3(x) = \frac{1 + x^3}{1 + x^2},$$

is continuous at 1.

Unit 3

3.1 Which of the following subsets of $\mathbf{R}^2$ are open in $\{\mathbf{R}^2, d\}$, where d is the Euclidean metric?

(i) $\mathbf{R}^2$

(ii) $\mathbf{R} \times \{0\}$

(iii) $\mathbf{R}^2 - \{(2, 3), (-2, -3)\}$

(iv) $\mathbf{R}^2 - \{(x, 0): x > 0\}$

(v) $\mathbf{R}^2 - \{(x, 0): x \geq 0\}$

(vi) The set of all points with exactly one irrational coordinate

3.2 For each of the following functions $d: \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$, explain why d is not a metric on $\mathbf{R}$.

(i) $d(x, y) = \big||x| - |y|\big|, \quad x, y \in \mathbf{R}$

(ii) $$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 2|x| + 3|y|, & \text{if } x \neq y, \end{cases} \qquad x, y \in \mathbf{R}$$

(iii) $d(x, y) = |x - y|^3, \quad x, y \in \mathbf{R}$

3.3 Let $M_1 = \{A_1, d_1\}$ and $M_2 = \{A_2, d_2\}$ be metric spaces, and let $f: A_1 \longrightarrow A_2$ be a function with the property that there exists a positive real number K such that

$$d_2(f(x), f(y)) \leq K\, d_1(x, y)$$

for all x, y in A_1. Prove that f is (d_1, d_2)-continuous.

3.4 Let $d: \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$ be defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ |x| + |y|, & \text{if } x \neq y. \end{cases}$$

(i) Show that d is a metric on $\mathbf{R}$.

(ii) Determine the following open balls with respect to the metric d:
(a) $B_1(0)$; (b) $B_1(1)$; (c) $B_2(1)$.

(iii) For each positive real number δ, find the open ball $B_\delta(0)$.

(iv) Show that the function $f: \mathbf{R} \longrightarrow \mathbf{R}$ defined by

$$f(x) = 1 + x, \quad x \in \mathbf{R},$$

is not (d, d)-continuous at 0.

3.5 Let $d: \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$ be the function given by

$$d(x, y) = \begin{cases} |x - y| + 1, & \text{if } x < 0 \leq y \text{ or } y < 0 \leq x, \\ |x - y|, & \text{otherwise.} \end{cases}$$

(i) Verify that d is a metric on $\mathbf{R}$.

(ii) Show that $d(x, 0) > 1$ for all x in $\mathbf{R}$ such that $x < 0$.

(iii) Write down the following open balls in the metric space $\{\mathbf{R}, d\}$:
(a) $B_1(0)$; (b) $B_1(-1)$.

(iv) Show that if $0 < r < a$, then $B_r(a) = (a - r, a + r)$.

(v) Show that $[0, \infty)$ is open in $\{\mathbf{R}, d\}$.

3.6 Let e be the metric on $\mathbf{R}^2$ defined by putting

$$e(x, y) = \begin{cases} 0, & \text{if } x = y, \\ d_2(x, y), & \text{if } x \neq y \text{ and } x, y \text{ are collinear with } \mathbf{0}, \\ d_2(\mathbf{0}, x) + d_2(\mathbf{0}, y), & \text{otherwise,} \end{cases}$$

where $\mathbf{0} = (0, 0)$ and d_2 is the Euclidean metric on $\mathbf{R}^2$. [This is the metric on $\mathbf{R}^2$ introduced in Example 2, *Unit 3*, page 70.]

Let

$$G = \{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 < 1\},$$
$$H = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \geq 0, x_2 \geq 0\},$$
$$K = \{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 = x_2^2\}.$$

Which of the following sets are open in $\{\mathbf{R}^2, e\}$?

(i) G (ii) H (iii) K
(iv) $H - \{\mathbf{0}\}$ (v) $K - \{\mathbf{0}\}$ (vi) $G \cup H$
(vii) $G \cup K$ (viii) $H - G$

Unit 4

4.1 Let $A = \{a, b, c, d\}$.

(i) Explain why

$$\mathcal{U} = \{\varnothing, \{a, b\}, \{b, c\}, \{c, d\}, A\}$$

is not a topology on A.

(ii) Find the coarsest topology $\mathcal{T}$ on A such that $\mathcal{U} \subset \mathcal{T}$.

Let $B = \{1, 2, 3\}$ and let $\mathcal{V}$ be the topology $\{\varnothing, \{1\}, \{2, 3\}, B\}$ on B.

(iii) Show that $f: A \longrightarrow B$ given by

$$f(a) = f(b) = 1, \quad f(c) = 2, \quad f(d) = 3$$

is $(\mathcal{T}, \mathcal{V})$-continuous.

(iv) Show that $g: B \longrightarrow A$ given by

$$g(1) = a, \quad g(2) = b, \quad g(3) = d$$

is not $(\mathcal{V}, \mathcal{T})$-continuous.

4.2 Let $\mathcal{T} = \{\varnothing, H, K, A\}$ be a topology on the set A, where H, K are distinct non-empty proper subsets of A. Prove that

$$H \cap K = \varnothing, \quad H \cup K = A,$$
or $H \subset K$,
or $K \subset H$.

4.3 (i) Let $\mathbf{N}$, $\mathbf{O}$ denote the sets of positive integers and odd positive integers respectively. For each positive integer n, let

$$V_n = \mathbf{O} \cup \{k \in \mathbf{N} : k \geq n\}.$$

(a) Prove that the collection $\mathcal{V}$ of subsets of $\mathbf{N}$ consisting of $\varnothing$ and the sets V_n for $n \in \mathbf{N}$ is a topology on $\mathbf{N}$.

(b) Determine the topology induced by $\mathcal{V}$ on $\{1, 2, 3\}$.

(ii) Let $A = \{a, b, c\}$ and let

$$\mathscr{B} = \{\{a\}, \{a, b\}, \{a, c\}\}.$$

(a) Show that $\mathscr{B}$ is a synthetic basis for A.

(b) Let $f: A \longrightarrow \mathbf{N}$ be given by

$$f(a) = 1, \quad f(b) = 2, \quad f(c) = 4.$$

Prove that f is $(\mathscr{T}, \mathscr{V})$-continuous, where $\mathscr{T}$ is the topology on A arising from $\mathscr{B}$, and $\mathscr{V}$ is the topology on $\mathbf{N}$ defined in part (i).

(c) State whether or not $g: A \longrightarrow \mathbf{N}$ given by

$$g(a) = 6, \quad g(b) = 1, \quad g(c) = 3$$

is $(\mathscr{T}, \mathscr{V})$-continuous, justifying your answer.

4.4 Let $\mathscr{T}$ be the collection of subsets of the set $\mathbf{Z}$ of integers defined as follows: $V \in \mathscr{T}$ if either $0 \notin V$ or $\mathbf{N} \subset V$, where $\mathbf{N}$ is the set of positive integers.

(i) Show that $\mathscr{T}$ is a topology on $\mathbf{Z}$.

(ii) Determine the topology induced by $\mathscr{T}$ on:

(a) $\{0, 1\}$; (b) $\{n \in \mathbf{Z}: n \leq 0\}$.

4.5 Let $T_1 = \{A, \mathscr{T}\}$, where $A = \{a, b, c\}$ and $\mathscr{T}$ is the topology

$$\{\varnothing, \{a\}, \{a, b\}, \{a, c\}, A\},$$

and let $T_2 = \{\mathbf{Z}, \mathscr{V}\}$, where $\mathbf{Z}$ is the set of all integers and $\mathscr{V}$ is the topology defined as follows: $V \in \mathscr{V}$ if either $0 \notin V$ or $1 \in V$.

(i) Prove that $f: A \longrightarrow \mathbf{Z}$ given by

$$f(a) = f(b) = 1, \quad f(c) = 0$$

is $(\mathscr{T}, \mathscr{V})$-continuous.

(ii) Prove that $g: A \longrightarrow \mathbf{Z}$ given by

$$g(a) = 0, \quad g(b) = g(c) = 1$$

is not $(\mathscr{T}, \mathscr{V})$-continuous.

(iii) State whether or not $h: A \longrightarrow \mathbf{Z}$ given by

$$h(a) = -1, \quad h(b) = h(c) = 0$$

is $(\mathscr{T}, \mathscr{V})$-continuous, justifying your answer.

4.6 Let $\mathbf{N}$ denote the set of positive integers, and let $\mathscr{B} = (B_n)_{n \in N}$ be the collection of subsets of $\mathbf{N}$ defined as follows:

$$B_n = \begin{cases} \{n\}, & \text{if } n \text{ is even,} \\ \{n\} \cup \{m: m \text{ even and } m > n\}, & \text{if } n \text{ is odd.} \end{cases}$$

(i) Prove that $\mathscr{B}$ is a synthetic basis for $\mathbf{N}$.

(ii) For each of the following subsets of $\mathbf{N}$ state, justifying your answer, whether or not it is open in $\{\mathbf{N}, \mathscr{T}\}$, where $\mathscr{T}$ is the topology on $\mathbf{N}$ arising from $\mathscr{B}$:

(a) the set $\mathbf{E}$ of even positive integers;

(b) the set $\mathbf{O}$ of odd positive integers;

(c) the set $\mathbf{E} \cup \{3^r: r \in \mathbf{N}\}$.

4.7 (i) Let $\mathscr{D}$ denote the discrete topology on the set $\mathbf{N}$ of positive integers, and let $\{A, \mathscr{T}\}$ be a topological space. Show that $f: \{A, \mathscr{T}\} \longrightarrow \{\mathbf{N}, \mathscr{D}\}$ is continuous if and only if

$$f^{-1}(\{n\}) \in \mathscr{T} \quad \text{for each } n \in \mathbf{N}.$$

(ii) Let $A = [1, \infty)$, and for each $n \in \mathbf{N}$, let $B_n = [n, n+1)$.

(a) Show that $\mathscr{B} = \{B_n : n \in \mathbf{N}\}$ is a synthetic basis for A.

(b) Show that $f: A \longrightarrow \mathbf{N}$ given by

$$f(x) = \text{greatest integer not exceeding } x$$

is $(\mathscr{T}, \mathscr{D})$-continuous, where $\mathscr{T}$ is the topology on A arising from $\mathscr{B}$.

Unit 5

5.1 Let $A = \{a, b, c, d, e\}$, and let $\mathscr{T}$ be the topology on A with the sets

$$\{a\}, \{a, d\}, \{a, e\}, \{b, c\}$$

as basis.

(i) Find all the closed sets of the topological space $\{A, \mathscr{T}\}$.

(ii) Find the closure in $\{A, \mathscr{T}\}$ of each of the sets $\{a, b\}$, $\{b, c\}$ and $\{b, d, e\}$.

5.2 Let $A = (0, 1)$, and let $\mathscr{T}$ be the topology on A which consists of $\varnothing$, A and the intervals $\left(0, \frac{n}{n+1}\right)$ for $n = 1, 2, \ldots$.

(i) Write down the closed sets of $\{A, \mathscr{T}\}$.

(ii) Show that for x in A, x is a limit point of $\{\frac{1}{4}, \frac{1}{2}\}$ if and only if $x \neq \frac{1}{4}$.

(iii) Find the closure in $\{A, \mathscr{T}\}$ of $\{\frac{1}{4}, \frac{1}{2}\}$.

(iv) Show that $\{A, \mathscr{T}\}$ is not a Hausdorff space.

5.3 Let $A = (0, 2)$, and let $\mathscr{T}$ be the topology on A which consists of $\varnothing$, A and the intervals $\left(\frac{1}{n}, 2 - \frac{1}{n}\right)$ for $n = 2, 3, \ldots$.

(i) Prove that if H and K are non-empty closed sets of $\{A, \mathscr{T}\}$, then $H \cap K \neq \varnothing$.

(ii) Show that for x in A, x, is a limit point of $\{\frac{1}{2}, 1\}$ if and only if $x \neq 1$.

(iii) Find the closure in $\{A, \mathscr{T}\}$ of $\{\frac{1}{2}, 1\}$.

(iv) Find the closure in $\{A, \mathscr{T}\}$ of $\{\frac{1}{3}\}$.

(v) Is $\{A, \mathscr{T}\}$ a Hausdorff space? Give a reason for your answer.

5.4 Let A denote the interval $(0, 1)$ in $\mathbf{R}$, and let $\mathscr{T}$ be the topology on A which consists of $\varnothing$ and all intervals of the form (a, b), where $0 \leq a \leq \frac{1}{2} \leq b \leq 1$ and $a \neq b$. Let $H = [\frac{1}{4}, \frac{1}{3}]$.

(i) Prove that if $x \in A$ and $x \leq \frac{1}{3}$, then x is a limit point of H in $\{A, \mathscr{T}\}$.

(ii) Prove that if $x \in A$ and $x > \frac{1}{3}$, then x is not a limit point of H in $\{A, \mathscr{T}\}$.

(iii) Determine the closure of H in $\{A, \mathscr{T}\}$.

5.5 Let $\mathscr{T}$ be the topology on the set $\mathbf{N}$ of positive integers defined in Exercise 4.6 of this Exercise Booklet.

(i) Prove that the set $\mathbf{O}$ of odd positive integers is closed in $\{\mathbf{N}, \mathscr{T}\}$.

(ii) Find the closure in $\{\mathbf{N}, \mathscr{T}\}$ of the set $\mathbf{E}$ of even positive integers.

(iii) (a) Show that 3 is a limit point of $\{2,4\}$ in $\{\mathbf{N}, \mathscr{T}\}$.
(b) Find the closure of $\{2,4\}$ in $\{\mathbf{N}, \mathscr{T}\}$.

(iv) Decide whether or not $\{\mathbf{N}, \mathscr{T}\}$ is a Hausdorff space, and justify your answer.

Unit 6

6.1 Show that the subspace

$$H = \{(x_1, x_2): x_1 + x_2 \geq 0\}$$

of $\mathbf{R}^2$ with the usual topology is not compact, by writing down an appropriate open cover of H and showing that it does not have a finite subcover.

6.2 Let $A = [-1, 1]$, and let $\mathscr{T}$ be the topology on A defined as follows: if $V \subset A$, then $V \in \mathscr{T}$ if either $0 \notin V$ or $(-1, 1) \subset V$.

(i) Show that $\{A, \mathscr{T}\}$ is a compact space.

(ii) Show that $\{\frac{1}{3}, \frac{2}{3}\}$ is a compact set in $\{A, \mathscr{T}\}$ which is not closed.

(iii) Is $\{A, \mathscr{T}\}$ a Hausdorff space? Give a reason for your answer.

6.3 Let $A = [0, 1]$, and let $\mathscr{T}$ be the topology on A defined as follows: if $V \subset A$, then $V \in \mathscr{T}$ if either $0 \notin V$ or $[0, \frac{1}{2}) \subset V$. Show that $\{A, \mathscr{T}\}$ is not a compact space.

6.4 Let A be the set of non-negative integers, let $\mathscr{B}$ be the synthetic basis for A consisting of A and all sets of the form

$$B_{2m} = \{0\} \cup \{n \in A: n \text{ even and } n \geq 2m\},$$
$$B_{2m+1} = \{1\} \cup \{n \in A: n \text{ odd and } n \geq 2m+1\},$$

for all positive integers m, and let $\mathscr{T}$ be the topology on A arising from $\mathscr{B}$.

(i) Prove that if V is open in $\{A, \mathscr{T}\}$ and $2 \in V$, then $B_2 \subset V$.

(ii) Show that $\{A, \mathscr{T}\}$ is a compact space.

6.5 Let $\mathbf{N}$ be the set of positive integers. For each positive integer k, let

$$B_k = \{mk: m \in \mathbf{N}\} = \{k, 2k, 3k, \ldots\},$$

and let $\mathscr{T}$ be the topology on $\mathbf{N}$ which has the sets B_k, $k \in \mathbf{N}$, as a basis.

(i) Prove that $\{\mathbf{N}, \mathscr{T}\}$ is a compact space.

(ii) Is the set $\mathbf{O}$ of odd positive integers a compact subset of $\{\mathbf{N}, \mathscr{T}\}$? Give a reason for your answer.

(iii) Prove that the set $\mathbf{E}$ of even positive integers is a compact subset of $\{\mathbf{N}, \mathscr{T}\}$ which is not closed.

(iv) Deduce from part (iii) that $\{\mathbf{N}, \mathscr{T}\}$ is not a Hausdorff space.

6.6 For each of the following subsets of $\mathbf{R}^2$ with the usual topology, state whether or not it is (a) closed, (b) bounded, (c) compact. For each subset, justify your answer to (c).

$$H_1 = \{(x_1, x_2): |x_1| \leq 2, |x_2| \leq 2, x_1^2 + x_2^2 > 1\}$$
$$H_2 = \{(x_1, x_2): 0 \leq x_1 \leq x_2\}$$
$$H_3 = \{(x_1, x_2): 0 \leq x_2 \leq |x_1| \leq 1\}$$

6.7 Let e be the metric on $\mathbf{R}$ defined by

$$e(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \text{Max}\{|x|, |y|\}, & \text{if } x \neq y. \end{cases}$$

[The metric e was introduced in Example 1, *Unit 3*, page 69.]

(i) Prove that a subset V of $\mathbf{R}$ is open in the metric space $M = \{\mathbf{R}, e\}$ if and only if either $0 \notin V$ or $(-r, r) \subset V$ for some positive real number r.

(ii) Deduce that if H is a subset of $\mathbf{R}$ and $0 \notin H$, then H is closed in M if and only if there exists a positive real number r such that $|x| \geq r$ for all x in H.

(iii) Show that $[1, 2]$ is a closed bounded subset of the metric space M which is not compact.

(iv) Show that

$$\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$$

is a compact subset of the metric space M.

Unit 7

7.1 This question concerns the topological space $\{A, \mathscr{T}\}$, where

$$A = \{a, b, c, d, e\},$$
$$\mathscr{T} = \{\varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, A\}.$$

(i) Show that the topological space $\{A, \mathscr{T}\}$ is not connected.

(ii) Show that $\{b, d, e\}$ with the induced topology as a subspace of $\{A, \mathscr{T}\}$ is connected.

(iii) Find the components of the topological space $\{A, \mathscr{T}\}$.

7.2 For each positive integer n, let K_n be the line segment in $\mathbf{R}^2$ joining $\left(\frac{1}{n}, 0\right)$ to $\left(\frac{1}{n}, 1\right)$, let I be the line segment joining $(0, 0)$ to $(1, 0)$ and let

$$H = \{(0, 1)\} \cup I \cup \bigcup_{n=1}^{\infty} K_n.$$

Prove that the subspace H of $\mathbf{R}^2$ with the usual topology is connected.

7.3 For each positive integer n, let L_n be the line segment in $\mathbf{R}^2$ joining $(0, 0)$ to $\left(\frac{1}{n}, 1\right)$, let

$$J = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 = 0,\ 0 \leq x_2 < \tfrac{1}{4} \text{ or } \tfrac{3}{4} < x_2 \leq 1\}$$

and let

$$K = J \cup \bigcup_{n=1}^{\infty} L_n.$$

Prove that the subspace K of $\mathbf{R}^2$ with the usual topology is connected.

7.4 For each positive integer n, let K_n be the line segment in $\mathbf{R}^2$ joining $\left(0, \frac{1}{n}\right)$ to $\left(1, \frac{1}{n}\right)$ and let L_n be the line segment joining $(0, 0)$ to $\left(1, \frac{1}{n}\right)$. Also, let I be the line segment joining $(0, 0)$ to $(1, 0)$ and let J be the line segment joining $(0, 0)$ to $(0, 1)$. Show that no two of the subspaces A, B and C of $\mathbf{R}^2$ with the usual topology are homeomorphic to each other, where

$$A = I \cup \bigcup_{n=1}^{\infty} K_n, \quad B = \bigcup_{n=1}^{\infty} L_n, \quad C = J \cup \bigcup_{n=1}^{\infty} K_n.$$

Solutions

1.1 (i) We must show that if $x \in \bigcup_{i \in I}(A_i - B_i)$, then $x \in \left(\bigcup_{i \in I} A_i\right) - \left(\bigcap_{i \in I} B_i\right)$. Now,

$$\begin{aligned} x \in \bigcup_{i \in I}(A_i - B_i) &\Rightarrow \text{for some } i,\ x \in A_i - B_i \\ &\Rightarrow \text{for some } i,\ x \in A_i \text{ and } x \notin B_i \\ &\Rightarrow x \in \bigcup_{i \in I} A_i \text{ and } x \notin \bigcap_{i \in I} B_i \\ &\Rightarrow x \in \left(\bigcup_{i \in I} A_i\right) - \left(\bigcap_{i \in I} B_i\right). \end{aligned}$$

Thus

$$\bigcup_{i \in I}(A_i - B_i) \subset \left(\bigcup_{i \in I} A_i\right) - \left(\bigcap_{i \in I} B_i\right).$$

There is an alternative proof based on the De Morgan formula. If we put $A = \bigcup_i A_i$, then $A_i - B_i \subset A - B_i$ for each i in I, so that

$$\bigcup_{i \in I}(A_i - B_i) \subset \bigcup_{i \in I}(A - B_i) = A - \left(\bigcap_{i \in I} B_i\right).$$

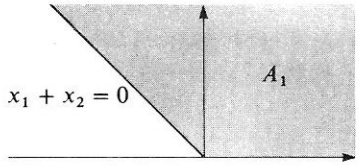
(ii) Let $A_1 = \{1,2\}$, $A_2 = \{3,4\}$, $B_1 = \{2\}$, $B_2 = \{4\}$. Then

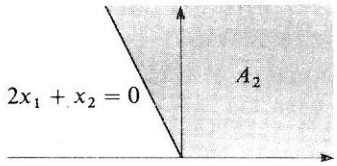
$$(A_1 - B_1) \cup (A_2 - B_2) = \{1\} \cup \{3\} = \{1,3\}.$$

But $A_1 \cup A_2 = \{1,2,3,4\}$ and $B_1 \cap B_2 = \varnothing$, so that

$$(A_1 \cup A_2) - (B_1 \cap B_2) = \{1,2,3,4\}.$$

1.2 (i)





(ii) Put $A = \bigcap_{n=1}^{\infty} A_n$ and $B = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \geq 0, x_2 \geq 0\}$. The assertion to be proved is that $A = B$. To establish this we must show that $A \subset B$ and that $B \subset A$.

The inclusion $B \subset A$ is readily established, because $B \subset A_n$ for every n and so $B \subset \bigcap_{n=1}^{\infty} A_n = A$. To show that $A \subset B$, we must show that if $x \in A$, then $x \in B$. But this is the case if and only if $x \notin B \Rightarrow x \notin A$, and it is this implication which we prove.

Suppose $x = (x_1, x_2) \notin B$. Then either $x_2 < 0$ or $x_2 \geq 0$, $x_1 < 0$.

If $x_2 < 0$, then $(x_1, x_2) \notin A_1$ so that $(x_1, x_2) \notin \bigcap_{n=1}^{\infty} A_n = A$.

If $x_2 \geq 0$, $x_1 < 0$, then, since $-x_1 > 0$, there exists a positive integer m such that $m(-x_1) > x_2$ and hence $mx_1 + x_2 < 0$; then $(x_1, x_2) \notin A_m$ and so $(x_1, x_2) \notin \bigcap_{n=1}^{\infty} A_n = A$.

This completes the proof that $A = B$.

1.3 (i) We recall that if $h\colon A \longrightarrow B$ is a function and D is a subset of the set B, then $h^{-1}(D)$ is the subset of the set A defined as follows:

$$h^{-1}(D) = \{x \in A\colon h(x) \in D\}.$$

(a) $f^{-1}([1,4]) = \{(x_1, x_2) \in \mathbf{R}^2\colon f(x_1, x_2) \in [1,4]\}$
$= \{(x_1, x_2) \in \mathbf{R}^2\colon 1 \leq x_1^2 + x_2^2 \leq 4\}$

(b) $f^{-1}(\{-4\}) = \{(x_1, x_2) \in \mathbf{R}^2\colon x_1^2 + x_2^2 = -4\}$
$= \varnothing$

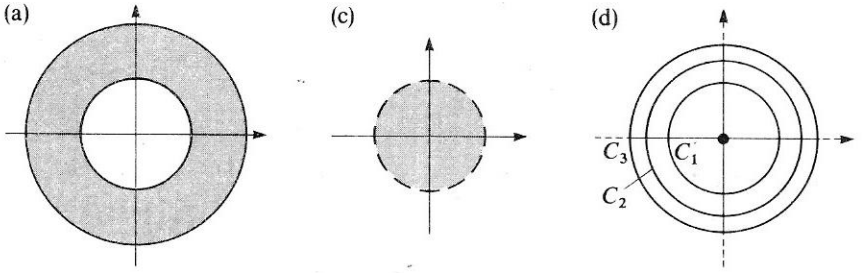
(c) $f^{-1}([-4,1)) = \{(x_1, x_2) \in \mathbf{R}^2\colon f(x_1, x_2) \in [-4,1)\}$
$= \{(x_1, x_2) \in \mathbf{R}^2\colon -4 \leq x_1^2 + x_2^2 < 1\}$
$= \{(x_1, x_2) \in \mathbf{R}^2\colon x_1^2 + x_2^2 < 1\}$

(d) $f^{-1}(\mathbf{Z}) = \{(x_1, x_2) \in \mathbf{R}^2\colon f(x_1, x_2) \in \mathbf{Z}\}$

$$= \{(0,0)\} \cup \bigcup_{n=1}^{\infty} C_n,$$

where for each positive integer n,

$$C_n = \{(x_1, x_2) \in \mathbf{R}^2\colon x_1^2 + x_2^2 = n\}.$$



(ii) (a) If $(x_1, x_2) \in [0,1] \times [0,1]$, then $0 \leq x_1^2 + x_2^2 \leq 2$ and $0 \leq x_1 x_2 \leq 1$. Thus $g([0,1] \times [0,1]) = [0,2] \times [0,1]$. Erratum.

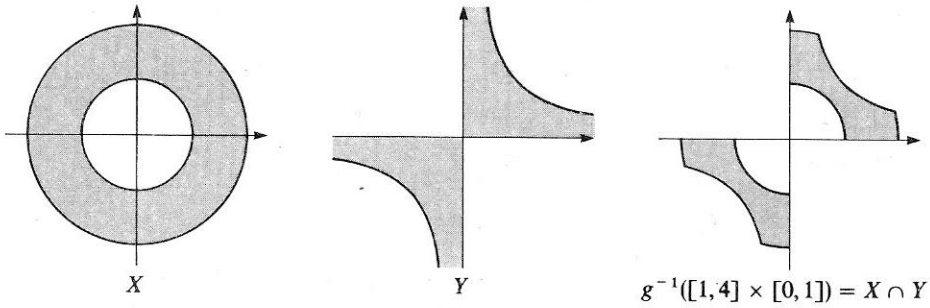
(b) $g^{-1}([1,4] \times [0,1]) = \{(x_1, x_2) \in \mathbf{R}^2\colon 1 \leq x_1^2 + x_2^2 \leq 4, 0 \leq x_1 x_2 \leq 1\}$
$= X \cap Y,$

where

$$X = \{(x_1, x_2) \in \mathbf{R}^2\colon 1 \leq x_1^2 + x_2^2 \leq 4\}$$

and

$$Y = \{(x_1, x_2) \in \mathbf{R}^2\colon 0 \leq x_1 x_2 \leq 1\}.$$



1.4 For each positive integer n,

$$f^{-1}(A_n) = \{x \in \mathbf{R}: f(x) \in A_n\}$$
$$= \left\{x \in \mathbf{R}: 0 \leq 1 - x^2 \leq 1 - \frac{1}{n^2}\right\}.$$

Now,

$$0 \leq 1 - x^2 \leq 1 - \frac{1}{n^2} \quad \Leftrightarrow \quad \frac{1}{n^2} \leq x^2 \leq 1$$
$$\Leftrightarrow \quad -1 \leq x \leq -\frac{1}{n} \quad \text{or} \quad \frac{1}{n} \leq x \leq 1$$
$$\Leftrightarrow \quad x \in \left[-1, -\frac{1}{n}\right] \quad \text{or} \quad x \in \left[\frac{1}{n}, 1\right].$$

Thus

$$f^{-1}(A_n) = \left[-1, -\frac{1}{n}\right] \cup \left[\frac{1}{n}, 1\right].$$

Let us put

$$B = [-1, 1] - \{0\} = [-1, 0) \cup (0, 1].$$

It is easy to see that if $x \in f^{-1}(A_n)$ for some n, then $x \in B$, so that

$$\bigcup_{n=1}^{\infty} f^{-1}(A_n) \subset B.$$

Conversely, if $x \in B$, then $0 < |x| \leq 1$. So there exists a positive integer m such that $\frac{1}{m} \leq |x|$ and hence $-1 \leq x \leq -\frac{1}{m}$ or $\frac{1}{m} \leq x \leq 1$, from which we see that $x \in f^{-1}(A_m) \subset \bigcup_{n=1}^{\infty} f^{-1}(A_n)$. Therefore

$$\bigcup_{n=1}^{\infty} f^{-1}(A_n) = B = [-1, 0) \cup (0, 1].$$

An alternative solution can be based on the result of SAQ 15(c), *Unit 1*, page 24. This tells us that

$$\bigcup_{n=1}^{\infty} f^{-1}(A_n) = f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right).$$

If $C = [0, 1)$, then $A_n \subset C$ for each positive integer n, so that $\bigcup_{n=1}^{\infty} A_n \subset C$. But if $x \in C$, then $x < 1$, so that $1 - x > 0$. There exists a positive integer m such that $\frac{1}{m^2} \leq 1 - x$, so that $x \leq 1 - \frac{1}{m^2}$. Also, $x \geq 0$ and so $x \in \left[0, 1 - \frac{1}{m^2}\right] = A_m \subset \bigcup_{n=1}^{\infty} A_n$. Therefore

$$\bigcup_{n=1}^{\infty} A_n = C = [0, 1),$$

so that

$$\bigcup_{n=1}^{\infty} f^{-1}(A_n) = f^{-1}(C) = \{x \in \mathbf{R}: 0 \leq 1 - x^2 < 1\}.$$

But

$$0 \leq 1 - x^2 < 1 \quad \Leftrightarrow \quad 0 < x^2 \leq 1 \quad \Leftrightarrow \quad -1 \leq x < 0 \text{ or } 0 < x \leq 1.$$

Hence

$$\bigcup_{n=1}^{\infty} f^{-1}(A_n) = [-1, 0) \cup (0, 1].$$

1.5 (i) It must be shown that each element of $(A \times C) \cup (B \times D)$ is an element of $(A \cup B) \times (C \cup D)$. We recall that if X and Y are sets, then the elements of $X \times Y$ are the ordered pairs (x, y), where $x \in X$ and $y \in Y$. Now

$$\begin{aligned}(x,y) \in (A \times C) \cup (B \times D) &\Rightarrow (x,y) \in A \times C \quad \text{or} \quad (x,y) \in B \times D \\ &\Rightarrow (x \in A \text{ and } y \in C) \quad \text{or} \quad (x \in B \text{ and } y \in D) \\ &\Rightarrow (x \in A \text{ or } x \in B) \quad \text{and} \quad (y \in C \text{ or } y \in D) \\ &\Rightarrow x \in A \cup B \quad \text{and} \quad y \in C \cup D \\ &\Rightarrow (x,y) \in (A \cup B) \times (C \cup D).\end{aligned}$$

Thus

$$(A \times C) \cup (B \times D) \subset (A \cup B) \times (C \cup D).$$

(ii) Take, for example,

$$A = C = \{0\}, \quad B = D = \{1\}.$$

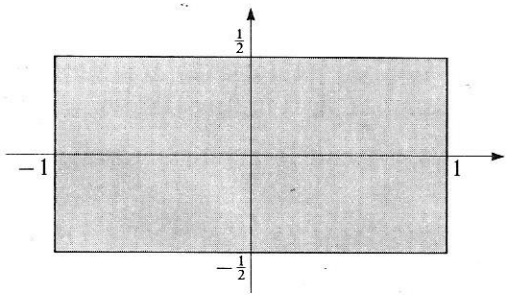
Then

$$(A \times C) \cup (B \times D) = \{(0,0), (1,1)\},$$

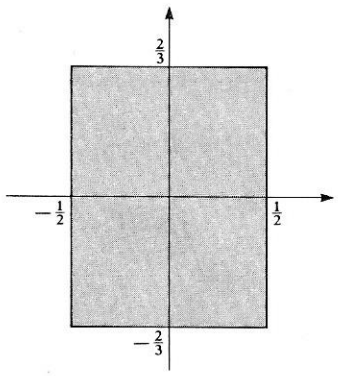
whilst

$$\begin{aligned}(A \cup B) \times (C \cup D) &= \{0,1\} \times \{0,1\} \\ &= \{(0,0),(0,1),(1,0),(1,1)\}.\end{aligned}$$

1.6 (i) $A_1 \times B_1$



$A_2 \times B_2$



(ii) Put $C = \{0\} \times [-\frac{1}{2}, \frac{1}{2}]$. We shall prove that

$$C = \bigcap_{n=1}^{\infty} (A_n \times B_n).$$

Since $C \subset A_n \times B_n$ for every n, it follows that

$$C \subset \bigcap_{n=1}^{\infty} (A_n \times B_n).$$

Now let $(a_1, a_2) \in \bigcap_{n=1}^{\infty} (A_n \times B_n)$. Then $(a_1, a_2) \in A_n \times B_n$ for all n, so that $a_1 \in A_n$, $a_2 \in B_n$ for all n.

Thus $|a_1| \leq \dfrac{1}{n}$ for every n, so that $a_1 = 0$; and $a_2 \in B_1 = [-\frac{1}{2}, \frac{1}{2}]$, and hence $-\frac{1}{2} \leq a_2 \leq \frac{1}{2}$. Thus $(a_1, a_2) \in C$.

Hence $\bigcap_{n=1}^{\infty} (A_n \times B_n) \subset C$. Therefore

$$C = \bigcap_{n=1}^{\infty} (A_n \times B_n).$$

2.1 (i) Put $k = \text{Max}\{\sup A, \sup B\}$. If $x \in A \cup B$, then either $x \in A$ or $x \in B$. If $x \in A$, then $x \leq \sup A \leq k$, and if $x \in B$, then $x \leq \sup B \leq k$. Thus k is an upper bound of $A \cup B$.

Since $A \cap B \subset A \cup B$, if follows that $A \cap B$ is also bounded above.

(ii) Suppose that $\sup B \leq \sup A$, so that $k = \text{Max}\{\sup A, \sup B\} = \sup A$. It was proved in part (i) that k is an upper bound of $A \cup B$. If $k' < k$, then there exists $x \in A \subset A \cup B$ such that $x > k'$.

Thus $k = \sup(A \cup B)$. A similar argument works if $\sup A \leq \sup B$.

(iii) If $x \in A \cap B$, then $x \in A$ so that $x \leq \sup A$, and $x \in B$ so that $x \leq \sup B$. Thus $x \leq \text{Min}\{\sup A, \sup B\}$ for all $x \in A \cap B$, and so $\text{Min}\{\sup A, \sup B\}$ is an upper bound of $A \cap B$. Therefore

$$\sup(A \cap B) \leq \text{Min}\{\sup A, \sup B\}.$$

(iv) Let

$$A = \{1, 2\}, \quad B = \{1, 3\}.$$

Then $A \cap B = \{1\}$, but $\text{Min}\{\sup A, \sup B\} = 2$.

2.2 (i) Suppose first that $\inf A = \alpha$ and let $\varepsilon > 0$ be given. Then $\alpha + \varepsilon > \alpha$, so that $\alpha + \varepsilon$ is not a lower bound of A. Hence there exists $a \in A$ such that $\alpha \leq a < \alpha + \varepsilon$.

Now suppose that for each $\varepsilon > 0$ we can find $a \in A$ such that $\alpha \leq a < \alpha + \varepsilon$. It follows that no number greater than α is a lower bound of A. Since α is a lower bound of A, it follows that $\alpha = \inf A$.

(ii) (a) Let $\alpha = \inf A$, $\beta = \inf B$. Then $\alpha \leq a$ and $\beta \leq b$ for all $a \in A$, $b \in B$. Hence

$$3a + 2b \geq 3\alpha + 2\beta$$

for all $a \in A$, $b \in B$. Thus $3\alpha + 2\beta$ is a lower bound of C.

(b) Since we know that $3\alpha + 2\beta$ is a lower bound of C, we can use the result of part (i). Let $\varepsilon > 0$ be given. Since $\alpha = \inf A$, $\beta = \inf B$, it follows from part (i) that there exist $a \in A$, $b \in B$ such that

$$\alpha \leq a < \alpha + \frac{\varepsilon}{5}, \quad \beta \leq b < \beta + \frac{\varepsilon}{5}.$$

Then

$$3\alpha + 2\beta \leq 3a + 2b < 3\left(\alpha + \frac{\varepsilon}{5}\right) + 2\left(\beta + \frac{\varepsilon}{5}\right)$$
$$= (3\alpha + 2\beta) + \varepsilon.$$

If we take $c = 3a + 2b$, then $c \in C$ and

$$3\alpha + 2\beta \leq c < (3\alpha + 2\beta) + \varepsilon.$$

It now follows from part (i) that

$$\inf C = 3\alpha + 2\beta.$$

2.3 (i) To say the sequence (s_n) is monotonic increasing means that $s_n \leq s_{n+1}$ for every positive integer n. To establish this, we note that

$$s_{n+1} - s_n = \frac{2n+1}{3n+4} - \frac{2n-1}{3n+1} = \frac{5}{(3n+1)(3n+4)} > 0,$$

so that $s_n < s_{n+1}$ for every positive integer n.

(ii) To establish this inequality we note that

$$\frac{2}{3} - s_n = \frac{2}{3} - \frac{2n-1}{3n+1} = \frac{5}{9n+3}.$$

Now $9n + 3 > 9n$, so that

$$0 < \frac{5}{9n+3} < \frac{5}{9n} < \frac{1}{n}.$$

Hence

$$0 < \frac{2}{3} - s_n < \frac{1}{n},$$

so that

$$\frac{2}{3} - \frac{1}{n} < s_n < \frac{2}{3}.$$

(iii) We must show that, given $\varepsilon > 0$, we can find a positive integer N such that $\left|s_n - \frac{2}{3}\right| < \varepsilon$ for all $n \geq N$. From part (ii),

$$\left|s_n - \frac{2}{3}\right| = \frac{2}{3} - s_n < \frac{1}{n}.$$

Let $\varepsilon > 0$ be given, and choose a positive integer N such that $\frac{1}{N} < \varepsilon$. Then for all $n \geq N$,

$$\left|s_n - \frac{2}{3}\right| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon,$$

as required.

2.4 (i) Since

$$f(x) - f(1) = x^2 - 1 = (x+1)(x-1),$$

we have

$$|f(x) - f(1)| = |x+1||x-1|.$$

Since

$$|x+1| = |(x-1)+2| \leq |x-1| + 2,$$

it follows that if $|x-1| < 1$ then

$$|f(x) - f(1)| < 3|x-1|. \qquad (*)$$

Let $\delta = \text{Min}\{1, \varepsilon/3\}$. If $|x-1| < \delta$, then $|x-1| < 1$, so that $(*)$ holds. But $|x-1| < \varepsilon/3$, so that

$$|f(x) - f(1)| < 3 \times \varepsilon/3 = \varepsilon.$$

(ii) Since

$$g(x) - g(1) = x^3 - 1 = (x^2 + x + 1)(x-1),$$

we have

$$|g(x) - g(1)| = |x^2 + x + 1||x-1|.$$

Also, since

$$|x^2 + x + 1| = |(x-1)^2 + 3(x-1) + 3|$$
$$\leq |x-1|^2 + 3|x-1| + 3,$$

it follows that $|x^2 + x + 1| < 7$ if $|x - 1| < 1$. Thus if $|x - 1| < 1$, then

$$|g(x) - g(1)| < 7|x - 1|. \qquad (**)$$

Given $\varepsilon > 0$, let $\delta = \text{Min}\{1, \varepsilon/7\}$. If $|x - 1| < \delta$, then $|x - 1| < 1$, so that $(**)$ holds. But $|x - 1| < \varepsilon/7$, and hence

$$|g(x) - g(1)| < 7 \times \varepsilon/7 = \varepsilon.$$

(iii) It follows from parts (i) and (ii) that the functions f and g are continuous at 1. But

$$h_1 = f + g \quad \text{and} \quad h_2 = fg.$$

Moreover, $f(x) \neq 0$ for all $x \in \mathbf{R}$, and

$$h_3 = \frac{g}{f} = g \times \frac{1}{f}.$$

Thus it follows from the results of SAQ 20, *Unit 1*, page 59, parts (iii), (iv) and (v) respectively, that each of the functions h_1, h_2 and h_3 is continuous at 1.

3.1 We recall that a subset H of $\mathbf{R}^2$ is open in $\{\mathbf{R}^2, d\}$, where d is the Euclidean metric, if and only if for each a in H there is some $r > 0$ such that the open disc $B_r(a)$, with centre a and radius r, is contained in H. In the following answers, for those sets H which are not open, the points for which there is no such disc contained in H are identified. Of course, to show that such an H is not open, giving just one of these points is enough.

(i) OPEN.

(ii) The set $H = \mathbf{R} \times \{0\}$ is NOT OPEN. For each a in H, there is no $r > 0$ such that $B_r(a) \subset H$.

(iii) OPEN.

(iv) The set $H = \mathbf{R}^2 - \{(x, 0): x > 0\}$ is NOT OPEN, for $\mathbf{0} = (0, 0) \in H$ and there is no $r > 0$ such that $B_r(\mathbf{0}) \subset H$.

(v) OPEN.

(vi) The set H of all points with exactly one irrational coordinate is NOT OPEN. For each a in H, there is no $r > 0$ such that $B_r(a) \subset H$.

3.2 A function $d: \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$ is a metric on $\mathbf{R}$ if and only if the conditions (M1), (M2) and (M3) of **S** Definition 2.1.2, page 21, are satisfied. For each of the given functions, one of the conditions fails to be satisfied.

(i) (M1) fails; for example, $d(1, -1) = 0$.

(ii) (M2) fails; for example, $d(1, 0) = 2$, $d(0, 1) = 3$.

(iii) (M3) fails; for example, $d(-1, 1) = 8$, $d(-1, 0) = 1 = d(0, 1)$, so it is not the case that $d(-1, 1) \leq d(-1, 0) + d(0, 1)$.

3.3 To prove that $f: A_1 \longrightarrow A_2$ is (d_1, d_2)-continuous, we must show that it is (d_1, d_2)-continuous at each point of A_1; and to show that f is (d_1, d_2)-continuous at a in A_1, we must show that given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_1(x, a) < \delta \quad \Rightarrow \quad d_2(f(x), f(a)) < \varepsilon.$$

Let $a \in A_1$ and let $\varepsilon > 0$ be given. If $\delta = \varepsilon/K$, then $\delta > 0$ and $d_1(x, a) < \delta$ implies that

$$d_2(f(x), f(a)) \leq K\, d_1(x, a) < K\delta = \varepsilon.$$

Thus f is (d_1, d_2)-continuous at a. Since this is true for all a in A_1, it follows that f is (d_1, d_2)-continuous.

3.4 (i) To show that d is a metric on $\mathbf{R}$, we must check that the conditions (M1), (M2) and (M3) are satisfied.

(M1) If $x = y$, then $d(x, y) = 0$ by definition. If $x \neq y$, then either $x \neq 0$ or $y \neq 0$, so that either $|x| > 0$ or $|y| > 0$ and hence $d(x, y) > 0$.

(M2) This condition is obviously satisfied.

(M3) If follows from Note (i), *Unit 3*, page 68 that it is enough to show that if x, y and z are all different, then

$$d(x, z) \leq d(x, y) + d(y, z).$$

But if x, y and z are all different, then

$$\begin{aligned} d(x, y) + d(y, z) &= |x| + 2|y| + |z| \\ &\geq |x| + |z| = d(x, z), \end{aligned}$$

since $|y| \geq 0$.

Since d satisfies (M1), (M2) and (M3), it is a metric on $\mathbf{R}$.

(ii) (a) Since $d(0, y) = |y|$ for all y, we see that

$$B_1(0) = \{y \in \mathbf{R}: d(0, y) < 1\} = \{y \in \mathbf{R}: |y| < 1\} = (-1, 1).$$

(b) If $y \neq 1$, then $d(1, y) = 1 + |y|$. Thus there in no y such that $y \neq 1$ and $d(1, y) < 1$. Of course, if $y = 1$, then $d(1, y) = 0 < 1$. Hence

$$B_1(1) = \{y \in \mathbf{R}: d(1, y) < 1\} = \{1\}.$$

(c) Since $d(1, y) = 1 + |y|$ if $y \neq 1$, we see that

$$\begin{aligned} B_2(1) &= \{1\} \cup \{y \in \mathbf{R}: y \neq 1, 1 + |y| < 2\} \\ &= \{1\} \cup \{y \in \mathbf{R}: |y| < 1\} \\ &= (-1, 1]. \end{aligned}$$

(iii) Following a similar argument to that of part (ii)(a), we see that if $\delta > 0$, then

$$B_\delta(0) = \{y \in \mathbf{R}: |y| < \delta\} = (-\delta, \delta).$$

(iv) We shall prove this result by a contradiction argument. Suppose that f is (d, d)-continuous at 0. Since $f(0) = 1$, then for each $\varepsilon > 0$ there exists $\delta > 0$ such that $f(B_\delta(0)) \subset B_\varepsilon(1)$. Thus in particular there exists $\delta > 0$ such that $f(x) \in B_1(1)$ if $x \in B_\delta(0)$. Thus, on noting the results of parts (ii)(b) and (iii), we that there exists $\delta > 0$ such that $f(x) = 1$ if $x \in (-\delta, \delta)$. Since $f(\frac{1}{2}\delta) = 1 + \frac{1}{2}\delta \neq 1$, we have a contradiction. Hence f is not (d, d)-continuous at 0.

3.5 (i) We must check that the conditions (M1), (M2) and (M3) are satisfied.

(M1) $d(x, y) \geq |x - y|$ so that $d(x, y) \geq 0$ for all x, y and $d(x, y) > 0$ if $x \neq y$. Also, $d(x, x) = 0$ for all x in $\mathbf{R}$.

(M2) Since $|x - y| = |y - x|$, it follows that $d(y, x) = d(x, y)$ for all x, $y \in \mathbf{R}$.

(M3) Let x, y and z be distinct real numbers. Three cases must be considered.

(a) $x < 0 \leq z$

Then $d(x, z) = |x - z| + 1$.
If $y < 0$, then $d(x, y) = |x - y|$ and $d(y, z) = |y - z| + 1$.
If $y \geq 0$, then $d(x, y) = |x - y| + 1$ and $d(y, z) = |y - z|$. Since

$$|x - z| \leq |x - y| + |y - z|,$$

it follows that

$$d(x, z) \leq d(x, y) + d(y, z).$$

(b) $z < 0 \leq x$

Then a similar argument to (a) shows that

$$d(x, z) \leq d(x, y) + d(y, z).$$

(c) Neither (a) nor (b).

Then

$$d(x,z) = |x - z| \leq |x - y| + |y - z|$$
$$\leq d(x,y) + d(y,z).$$

Since d satisfies (M1), (M2) and (M3), it is a metric on $\mathbf{R}$.

(ii) If $x < 0$ then, by definition, $d(x,0) = |x| + 1$.

Also, $|x| > 0$, so $d(x,0) > 1$.

(iii) (a) We recall that

$$B_1(0) = \{x \in \mathbf{R}: d(x,0) < 1\}.$$

We have noted that $d(x,0) > 1$ if $x < 0$. Thus

$$B_1(0) = \{x \in \mathbf{R}: x \geq 0, d(x,0) < 1\}$$
$$= \{x \in \mathbf{R}: x \geq 0, |x| < 1\}$$
$$= [0,1).$$

(b) If $x \geq 0$, then $d(x,-1) = 1 + |x| \geq 1$. Thus

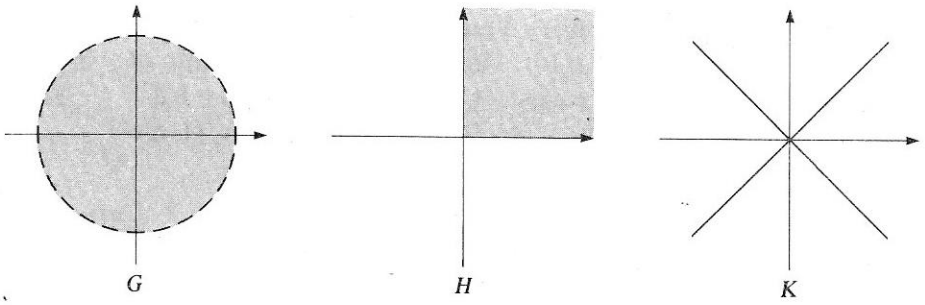
$$B_1(-1) = \{x \in \mathbf{R}: x < 0, |x + 1| < 1\}$$
$$= (-2,0).$$

(iv) If $x < 0$, then $d(x,a) = |a - x| + 1 = a - x + 1$ and $a - x + 1 > a > r$. If $x \geq 0$, then $d(x,a) = |x - a|$. Hence

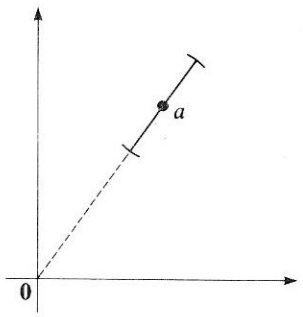
$$B_r(a) = \{x \in \mathbf{R}: x \geq 0 \text{ and } |x - a| < r\}$$
$$= (a - r, a + r).$$

(v) Put $V = [0,\infty) = \{x \in \mathbf{R}: x \geq 0\}$. To prove that V is open in $\{\mathbf{R}, d\}$ we must show that for each a in V, there exists $r > 0$ such that $B_r(a) \subset V$. It was proved in part (iii)(a) that $B_1(0) = [0,1) \subset V$. If $a > 0$, then by part (iv), $B_r(a) = (a - r, a + r) \subset V$ if $0 < r < a$. Thus V is open in $\{\mathbf{R}, d\}$.

3.6



Put $M = \{\mathbf{R}^2, e\}$. In Example 11, *Unit 3*, page 82, the open balls $B_r(a)$ in the metric space M were determined. Recall that for each $r > 0$, $B_r(0)$ is the (Euclidean) open disc in $\mathbf{R}^2$ with centre $\mathbf{0}$ and radius r, whilst if $a \neq \mathbf{0}$ and $0 < r \leq d_2(\mathbf{0}, a)$, then $B_r(a)$ consists of all points x on the line passing through $\mathbf{0}$ and a such that $d_2(x,a) < r$.



(i) Since $G = B_1(\mathbf{0})$, it is open in M.

(ii) The set H is not open in M, for there is no positive real number r such that $B_r(\mathbf{0}) \subset H$.

(iii) Likewise, K is not open in M, for there is no positive real number r such that $B_r(\mathbf{0}) \subset K$.

(iv) The set $H - \{\mathbf{0}\}$ is open in M, for if $a \in H - \{\mathbf{0}\}$ and $0 < r \leq d_2(\mathbf{0}, a)$, then $B_r(a) \subset H - \{\mathbf{0}\}$.

(v) The set $K - \{\mathbf{0}\}$ is open in M, by the same argument as employed in part (iv).

(vi) We observe that $G \cup H = G \cup (H - \{\mathbf{0}\})$. Since G and $H - \{\mathbf{0}\}$ are open in M and the union of open sets is open, $G \cup H$ is open in M.

(vii) The set $G \cup K$ is open in M, since $G \cup K = G \cup (K - \{\mathbf{0}\})$ and G, $K - \{\mathbf{0}\}$ are open in M.

(viii) The set $H - G$ is not open in M, for if $b = (0, 1)$, then $b \in H - G$ but there is no positive real number r such that $B_r(b) \subset H - G$.

4.1 To show that a collection $\mathscr{U}$ of subsets of A is a topology on A, we must verify that $\mathscr{U}$ satisfies the axioms (T1), (T2) and (T3) of S Definition 3.1.1, page 45.

(i) Since $\{a, b\} \in \mathscr{U}$, $\{b, c\} \in \mathscr{U}$ and $\{a, b\} \cap \{b, c\} = \{b\} \notin \mathscr{U}$, the collection $\mathscr{U}$ does not satisfy the axiom (T2) and so $\mathscr{U}$ is not a topology on A.

There are other violations of (T2) and (T3) which could equally well be cited to show that $\mathscr{U}$ is not a topology on A.

(ii) If $\mathscr{U}$ is a collection of subsets of a set A, then the coarsest topology $\mathscr{T}$ on A such that $\mathscr{U} \subset \mathscr{T}$ consists of all unions of finite intersections of members of $\mathscr{U}$. Thus in this particular case,

$$\mathscr{T} = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, A\}.$$

(iii) In general, if $\{A, \mathscr{T}\}$ and $\{B, \mathscr{V}\}$ are topological spaces then, in accordance with S Definition 3.1.2, page 45, to show that a function $f: A \longrightarrow B$ is $(\mathscr{T}, \mathscr{V})$-continuous, we must verify that if $V \in \mathscr{V}$, then $f^{-1}(V) \in \mathscr{T}$. In this case, we have

$$f^{-1}(\varnothing) = \varnothing,$$
$$f^{-1}(\{1\}) = \{a, b\},$$
$$f^{-1}(\{2, 3\}) = \{c, d\},$$
$$f^{-1}(B) = A.$$

Thus f is $(\mathscr{T}, \mathscr{V})$-continuous.

(iv) Since $\{b\} \in \mathscr{T}$ and $g^{-1}(\{b\}) = \{2\} \notin \mathscr{V}$, the function g is not $(\mathscr{V}, \mathscr{T})$-continuous.

4.2 We must consider two possibilities.

(a) $H \cap K = \varnothing$. Since (T3) is satisfied, $H \cup K \in \mathscr{T}$. But $H \cup K$ is distinct from $\varnothing$, H and K and hence $H \cup K = A$.

(b) $H \cap K \neq \varnothing$. Since (T2) is satisfied, $H \cap K \in \mathscr{T}$. Thus either $H \cap K = H$ or $H \cap K = K$. But if $H \cap K = H$ then $H \subset K$, and if $H \cap K = K$ then $K \subset H$.

Thus precisely one of the three stated conditions holds.

4.3 (i) (a) It must be shown that (T1), (T2) and (T3) are satisfied.

(T1) By definition, $\varnothing \in \mathscr{V}$. Also, $\mathbf{N} = V_1 \in \mathscr{V}$.

(T2) If $m, n \in \mathbf{N}$ and $m < n$, then $V_m \cap V_n = V_n \in \mathscr{V}$.

(T3) A non-empty collection of members of $\mathscr{V}$ is of the form $(V_j)_{j \in J}$, where J is a non-empty subset of $\mathbf{N}$. The non-empty subset J of $\mathbf{N}$ has a least element, k say. Since $V_j \subset V_k$ for each $j \in J$, we see that $\bigcup_{j \in J} V_j = V_k$ and hence $\bigcup_{j \in J} V_j \in \mathscr{V}$, as required.

Since (T1), (T2) and (T3) are satisfied, $\mathscr{V}$ is a topology on $\mathbf{N}$.

(b) In general, if $\{A, \mathscr{V}\}$ is a topological space and $B \subset A$, then, according to S Definition 3.4.1, page 51, the topology induced by $\mathscr{V}$ on B consists of all sets of the form $V \cap B$ for V in $\mathscr{V}$. In this case, putting $B = \{1, 2, 3\}$, we see that the non-empty members of the topology induced by $\mathscr{V}$ on B are the sets $V_n \cap B$ for $n \in \mathbf{N}$.

Since

$$V_1 \cap B = V_2 \cap B = B$$

and for $n > 2$

$$V_n \cap B = \{1, 3\},$$

the induced topology on B is

$$\{\varnothing, \{1, 3\}, B\}.$$

(ii) (a) By S Definition 3.2.2, page 49, a collection $\mathscr{B}$ of subsets of a set A is a synthetic basis for A if the conditions (B1) and (B2) are satisfied.

In this case we have $\{a, b\} \cup \{a, c\} = A$, so that (B1) is satisfied. Moreover,

$$\{a\} \cap \{a, b\} = \{a\} \cap \{a, c\} = \{a, b\} \cap \{a, c\} = \{a\},$$

so that (B2) is satisfied.

(b) Since

$$f^{-1}(V_r) = \begin{cases} A, & \text{if } r \leq 2, \\ \{a, c\}, & \text{if } 2 < r \leq 4, \\ \{a\}, & \text{if } r > 4, \end{cases}$$

it follows that $f^{-1}(V) \in \mathscr{B}$ for each $V \in \mathscr{V}$.

Since $\mathscr{B} \subset \mathscr{T}$, we have $f^{-1}(V) \in \mathscr{T}$ for each $V \in \mathscr{V}$, and so f is $(\mathscr{T}, \mathscr{V})$-continuous.

(c) The function g is not $(\mathscr{T}, \mathscr{V})$-continuous. To justify this statement, we must exhibit a member V of $\mathscr{V}$ such that $g^{-1}(V) \notin \mathscr{T}$.

Now

$$g^{-1}(V_7) = \{b, c\}$$

and $\{b, c\} \notin \mathscr{T}$ since $\{b, c\}$ cannot be expressed as a union of members of $\mathscr{B}$.

[If $\mathscr{B}$ is a synthetic basis for a set A, then the topology $\mathscr{T}$ on A arising from $\mathscr{B}$ consists of all unions of members of $\mathscr{B}$. In this case,

$$\mathscr{T} = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, A\}.]$$

4.4 (i) We must show that (T1), (T2) and (T3) are satisfied.

(T1) Since $0 \notin \varnothing$, $\varnothing \in \mathscr{T}$ and since $\mathbf{N} \subset \mathbf{Z}$, $\mathbf{Z} \in \mathscr{T}$.

(T2) Let $V_1, V_2 \in \mathscr{T}$. We must show that $V_1 \cap V_2 \in \mathscr{T}$. If $0 \notin V_1 \cap V_2$, then $V_1 \cap V_2 \in \mathscr{T}$. If $0 \in V_1 \cap V_2$, then $0 \in V_1$ and $0 \in V_2$. Since $V_1 \in \mathscr{T}$ and $0 \in V_1$, it follows that $\mathbf{N} \subset V_1$. Similarly, $\mathbf{N} \subset V_2$. Thus $\mathbf{N} \subset V_1 \cap V_2$, and so $V_1 \cap V_2 \in \mathscr{T}$.

(T3) Let $(V_i)_{i \in I}$ be a family of members of $\mathscr{T}$, and let $V = \bigcup_{i \in I} V_i$. We must show that $V \in \mathscr{T}$. If $0 \notin V$, then $V \in \mathscr{T}$. If $0 \in V$, then there exists j in I such that $0 \in V_j$. Since $V_j \in \mathscr{T}$ and $0 \in V_j$, it follows that $\mathbf{N} \subset V_j$. Hence $\mathbf{N} \subset V_j \subset V$, and so $V \in \mathscr{T}$.

(ii) (a) Put $H = \{0, 1\}$ and let $\mathscr{T}_H$ be the topology induced by $\mathscr{T}$ on H. If $W \in \mathscr{T}_H$ and $0 \in W$, then there exists $V \in \mathscr{T}$ such that $W = V \cap H$; but since $V \in \mathscr{T}$ and $0 \in V$, it follows that $\mathbf{N} \subset V$ and so $W = V \cap H = H$. Since $\{1\} = \{1\} \cap H$ and $\{1\} \in \mathscr{T}$, we see that

$$\mathscr{T}_H = \{\varnothing, \{1\}, H\}.$$

(b) Put $K = \{n \in \mathbf{Z} : n \leq 0\}$. If $A \subset K$, then $A \cup \mathbf{N} \in \mathscr{T}$. Since $(A \cup \mathbf{N}) \cap K = A$, we see that A belongs to the topology induced by $\mathscr{T}$ on K. Since every subset of K is an open set of the topology induced by $\mathscr{T}$ on K, it is the discrete topology on K.

4.5 (i) Let $V \in \mathscr{V}$. We must show that $f^{-1}(V) \in \mathscr{T}$. If $0 \notin V$, then

$$f^{-1}(V) = \begin{cases} \varnothing, & \text{if } 1 \notin V, \\ \{a, b\}, & \text{if } 1 \in V, \end{cases}$$

and so $f^{-1}(V) \in \mathscr{T}$.

If $0 \in V$, then $1 \in V$ since $V \in \mathscr{V}$, and so $f^{-1}(V) = A \in \mathscr{T}$. Thus f is $(\mathscr{T}, \mathscr{V})$-continuous.

(ii) To prove that g is not $(\mathscr{T}, \mathscr{V})$-continuous, we must find $V \in \mathscr{V}$ such that $g^{-1}(V) \notin \mathscr{T}$. But $\{1\} \in \mathscr{T}$ and $g^{-1}(\{1\}) = \{b, c\} \notin \mathscr{T}$. Thus g is not $(\mathscr{T}, \mathscr{V})$-continuous.

(iii) The function h is not $(\mathscr{T}, \mathscr{V})$-continuous, for $\{0, 1\} \in \mathscr{V}$, since $1 \in \{0, 1\}$, but $h^{-1}(\{0, 1\}) = \{b, c\} \notin \mathscr{T}$.

4.6 (i) To prove that $\mathscr{B}$ is a synthetic basis for $\mathbf{N}$, we appeal to Note (iii), *Unit 4*, page 103, and show that $\mathscr{B}$ satisfies (B1) and the condition (B2′) introduced in the note.

(B1) Since $\bigcup_{n \in \mathbf{N}} B_n = \mathbf{N}$, condition (B1) is satisfied.

(B2′) Let n, m be distinct positive integers. To show that (B2′) is satisfied, we must show that if $k \in B_n \cap B_m$, then there exists a positive integer p such that $k \in B_p \subset B_n \cap B_m$.

If n, m are both odd and $k \in B_n \cap B_m$, then k is even and $k > \text{Max}\{m, n\}$. Thus $B_k = \{k\}$ and $k \in B_k \subset B_n \cap B_m$.

If n, m are both even, then $B_n \cap B_m = \varnothing$, and there is nothing to prove.

Finally, if n is odd, m is even, and $B_n \cap B_m \neq \varnothing$, then $m > n$ and $B_n \cap B_m = B_m = \{m\}$; thus if $k \in B_n \cap B_m$, then $k = m$ and $k \in B_m \subset B_n \cap B_m$.

Thus (B2′) is satisfied.

Since $\mathscr{B}$ satisfies (B1) and (B2′), it is a synthetic basis for $\mathbf{N}$.

(ii) We recall that $V \in \mathscr{T}$ if and only if V is a union of members of $\mathscr{B}$. Equivalently, $V \in \mathscr{T}$ if and only if for each $k \in V$, there exists a positive integer n such that $k \in B_n \subset V$.

(a) If $k \in \mathbf{E}$, then $B_k = \{k\}$ and $k \in B_k \subset \mathbf{E}$. Thus $\mathbf{E}$ is an open set.

(b) The set $\mathbf{O}$ of odd positive integers is not an open set. Let $k \in \mathbf{O}$, then since k is odd, the only member of $\mathscr{B}$ which contains k is B_k, and

$$B_k = \{k\} \cup \{m \in \mathbf{E} : m > k\}.$$

Clearly, it is not true that $B_k \subset \mathbf{O}$.

(c) Put $H = \mathbf{E} \cup \{3^r : r \in \mathbf{N}\}$ and let $k \in H$. If $k \in \mathbf{E}$, then $B_k = \{k\}$ so $k \in B_k \subset H$. If $k = 3^r$, then

$$B_k = \{k\} \cup \{m \in \mathbf{E} : m > k\}$$

and again $k \in B_k \subset H$. Thus the set H is open.

4.7 (i) We must show that $f: \{A, \mathscr{T}\} \longrightarrow \{\mathbf{N}, \mathscr{D}\}$ is continuous if and only if the condition

$$(*) \qquad f^{-1}(\{n\}) \in \mathscr{T} \quad \text{for each } n \in \mathbf{N}$$

is satisfied.

Suppose first that f is continuous. Since $\{n\} \in \mathscr{D}$, it follows that $f^{-1}(\{n\}) \in \mathscr{T}$ for each n. Thus $(*)$ is satisfied.

Conversely, suppose $(*)$ is satisfied. If $V \in \mathscr{D}$, then $V = \bigcup_{n \in V} \{n\}$ so that

$$f^{-1}(V) = f^{-1}\left(\bigcup_{n \in V} \{n\}\right) = \bigcup_{n \in V} f^{-1}(\{n\}).$$

Since $f^{-1}(\{n\}) \in \mathscr{T}$ for each n in V, and $\mathscr{T}$ satisfies (T3), we see that $f^{-1}(V) \in \mathscr{T}$. Since $f^{-1}(V) \in \mathscr{T}$ for each $V \in \mathscr{D}$, f is continuous.

Thus $f: \{A, \mathscr{T}\} \longrightarrow \{\mathbf{N}, \mathscr{D}\}$ is continuous if and only if condition $(*)$ is satisfied.

A slicker proof of this result goes as follows. The collection $\{\{n\} : n \in \mathbf{N}\}$ is a basis for the discrete topology on $\mathbf{N}$. Hence, by S Application 3.2.5, page 50, $f: \{A, \mathscr{T}\} \longrightarrow \{\mathbf{N}, \mathscr{D}\}$ is continuous if and only if $f^{-1}(\{n\}) \in \mathscr{T}$ for each $n \in \mathbf{N}$.

(ii) (a) We must show that $\mathscr{B}$ satisfies (B1) and (B2). Since $\bigcup_{n \in \mathbf{N}} B_n = \mathbf{N}$, (B1) is satisfied. If n and m are distinct positive integers, then $B_n \cap B_m = \varnothing$, so (B2) is satisfied. Thus $\mathscr{B}$ is a synthetic basis for A.

(b) For each $n \in \mathbf{N}$,

$$f^{-1}(\{n\}) = B_n \in \mathscr{B} \subset \mathscr{T}.$$

It follows from the result of part (i) that f is $(\mathscr{T}, \mathscr{D})$-continuous.

5.1 (i) Since

$$\mathscr{T} = \{\varnothing, \{a\}, \{a,d\}, \{a,e\}, \{b,c\}, \{a,b,c\}, \{a,d,e\}, \{a,b,c,d\}, \{a,b,c,e\}, A\},$$

the set of closed sets of $\{A, \mathscr{T}\}$ is

$$\{A, \{b,c,d,e\}, \{b,c,e\}, \{b,c,d\}, \{a,d,e\}, \{d,e\}, \{b,c\}, \{e\}, \{d\}, \varnothing\}.$$

(ii) For each subset H of A, the closure of H in $\{A, \mathscr{T}\}$ is the smallest closed set containing H. Thus

$$Cl(\{a,b\}) = A,$$
$$Cl(\{b,c\}) = \{b,c\},$$
$$Cl(\{b,d,e\}) = \{b,c,d,e\}.$$

5.2 (i) The closed sets of $\{A, \mathscr{T}\}$ are $\varnothing$, A and the intervals

$$\left[\frac{n}{n+1}, 1\right) \text{ for } n = 1, 2, \ldots .$$

(ii) Put $H = \{\frac{1}{4}, \frac{1}{2}\}$.

Since $(0, \frac{1}{2}) \in \mathscr{T}$, $\frac{1}{4} \in (0, \frac{1}{2})$ and $(0, \frac{1}{2}) \cap H = \{\frac{1}{4}\}$, $\frac{1}{4}$ is not a limit point of H.

If $x \in A$, $x \neq \frac{1}{4}$, and G is an open set containing x, then $(0, \frac{1}{2}) \subset G$, so that $\frac{1}{4} \in G \cap H$. Thus x is a limit point of H.

(iii) Since the closure of H is the union of H and its set of limit points, it follows from part (ii) that $Cl(H) = A$.

An alternative argument can be based on the description of the closed sets of $\{A, \mathscr{T}\}$ obtained in part (i). The closure of H is the smallest closed set which contains H; since none of the intervals $\left[\frac{n}{n+1}, 1\right)$, $n = 1, 2, \ldots$, contains H, it follows that $Cl(H) = A$.

(iv) In a Hausdorff space, every finite set is closed. Since $\{\frac{1}{4}, \frac{1}{2}\}$ is not closed, $\{A, \mathscr{T}\}$ is not a Hausdorff space. (Of course, this, and similar exercises, can also be answered by reference to the definition of a Hausdorff space, but such an answer will take more effort than ours!)

5.3 (i) The non-empty closed sets of $\{A, \mathscr{T}\}$ are A and the sets

$$E_n = \left(0, \frac{1}{n}\right] \cup \left[2 - \frac{1}{n}, 2\right)$$

for each integer $n \geq 2$. It is enough to consider the case when H and K are E_n and E_m, where, without loss of generality, $m \geq n \geq 2$. Then $E_n \cap E_m = E_m \neq \varnothing$.

(ii) Put $H = \{\frac{1}{2}, 1\}$.

Since every non-empty open set contains 1, it follows that if $x \in A$ and $x \neq 1$, then x is a limit point of H.

But 1 is not a limit point of H, since $(\frac{1}{2}, \frac{3}{2})$ is an open set which contains 1 and no point of H other than 1.

(iii) $Cl(H) = A$, since the closure of H is the union of H and the set of limit points of H.

(iv) Since the closure of the singleton set $\{\frac{1}{3}\}$ is the smallest closed set which contains $\frac{1}{3}$, we see that

$$Cl(\{\tfrac{1}{3}\}) = E_3 = (0, \tfrac{1}{3}] \cup [\tfrac{5}{3}, 2).$$

(v) The space $\{A, \mathscr{T}\}$ is not a Hausdorff space; in a Hausdorff space, singleton sets are closed, but, from part (iv), $\{\frac{1}{3}\}$ is not closed in $\{A, \mathscr{T}\}$.

5.4 (i) Suppose that $x \leq \frac{1}{3}$, and let V be a proper open subset of $\{A, \mathscr{T}\}$ such that $x \in V$. Then $V = (a, b)$ where $a < x$ and $b \geq \frac{1}{2}$.

If $x < \frac{1}{3}$, then $\frac{1}{3} \in V \cap H$. If $x = \frac{1}{3}$, then there exists y such that

$$\text{Max}\{\tfrac{1}{4}, a\} < y < x = \tfrac{1}{3},$$

so that $y \in V \cap H$. In all cases there is an element different from x in $V \cap H$. Thus x is a limit point of H.

(ii) Let $x \in A$ with $x > \frac{1}{3}$. Since $x < 1$, $V = (\frac{1}{3}, 1)$ is an open set of $\{A, \mathscr{T}\}$ such that $x \in V$ and $V \cap H = \varnothing$. Hence x is not a limit point of H.

(iii) The closure of H in $\{A, \mathscr{T}\}$ is the union of H and its set of limit points. Thus $Cl(H) = (0, \frac{1}{3}]$.

5.5 (i) It was shown in the solution to Exercise 4.6(ii)(a) that the set $\mathbf{E}$ of even positive integers is open in $\{\mathbf{N}, \mathscr{T}\}$. As $\mathbf{O} = \mathbf{N} - \mathbf{E}$, it follows that the set $\mathbf{O}$ is closed in $\{\mathbf{N}, \mathscr{T}\}$.

(ii) Let n be an odd positive integer and let V be an open set such that $n \in V$. Then $B_n \subset V$, so that since

$$B_n = \{n\} \cup \{m \in \mathbf{E}: m > n\},$$

it follows that $V \cap E \neq \varnothing$. Hence $Cl(\mathbf{E}) = \mathbf{N}$.

(iii) Put $H = \{2, 4\}$.

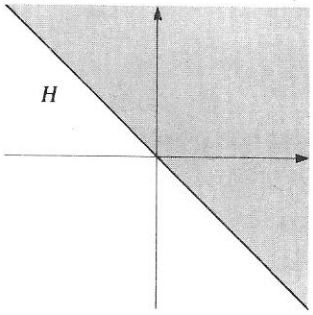
(a) If V is an open set of $\{\mathbf{N}, \mathscr{T}\}$ which contains 3, then $B_3 \subset V$. Since $4 \in B_3$, it follows that $V \cap H \neq \varnothing$. Thus 3 is a limit point of H.

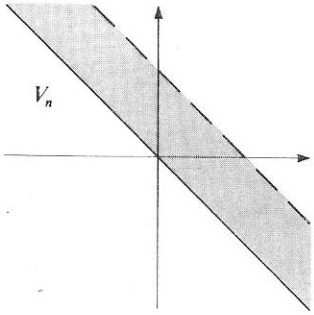
(b) Every open set which contains 1 contains B_1. Since $H \subset B_1$, it follows that 1 is a limit point of H. Thus 1 and 3 are limit points of H in $\{\mathbf{N}, \mathscr{T}\}$. If $n \in \mathbf{N}$ and $n > 4$, then $B_n \cap H = \varnothing$, so that n is not a limit point of H. Since the closure of H is the union of H and its set of limit points, it follows that $Cl(H) = \{1, 2, 3, 4\}$.

(iv) In a Hausdorff space, each finite set is closed. Since $H = \{2, 4\}$ is not closed in $\{\mathbf{N}, \mathscr{T}\}$, it follows that $\{\mathbf{N}, \mathscr{T}\}$ is not a Hausdorff space.

6.1 For each positive integer n, let

$$V_n = \{(x_1, x_2) \in H: x_1 + x_2 < n\}.$$





Since V_n is the intersection with H of the set

$$\{(x_1, x_2) \in \mathbf{R}^2: x_1 + x_2 < n\}$$

which is open in $\mathbf{R}^2$ with the usual topology, V_n is open in H for each positive integer n.

If $(x_1, x_2) \in H$, then there exists a positive integer m such that $x_1 + x_2 < m$ and so $(x_1, x_2) \in V_m$.

Thus $\bigcup_{n=1}^{\infty} V_n = H$, and so $\mathscr{V} = (V_n)_{n=1,2,\ldots}$ is an open cover of H.

The open cover $\mathscr{V}$ contains no finite subcover, for a finite subcollection of $\mathscr{V}$ is of the form $\{V_{n_1}, V_{n_2}, \ldots, V_{n_k}\}$, where $n_1 < n_2 < \ldots < n_k$ and

$$V_{n_1} \cup V_{n_2} \cup \ldots \cup V_{n_k} = V_{n_k} \neq H.$$

Since $\mathscr{V}$ is an open cover of H which contains no finite subcover, the topological space H is not compact.

6.2 (i) We must show that each open cover of $\{A, \mathscr{T}\}$ contains a finite subcover. Let $\mathscr{V}$ be an open cover of $\{A, \mathscr{T}\}$. There exists $V \in \mathscr{V}$ such that $0 \in V$. Since $0 \in V$ and $V \in \mathscr{T}$, it follows that $(-1, 1) \subset V$. Also, since $\mathscr{V}$ is a cover, there exist members V', V'' of $\mathscr{V}$ such that $1 \in V'$ and $-1 \in V''$ Then

$$V \cup V' \cup V'' = A.$$

Thus $\{V, V', V''\}$ is a finite subcover of $\mathscr{V}$. Therefore $\{A, \mathscr{T}\}$ is a compact space.

(ii) Any topological space containing only finitely many points is compact. Thus $B = \{\frac{1}{3}, \frac{2}{3}\}$ is a compact set in $\{A, \mathscr{T}\}$. But B is not closed in $\{A, \mathscr{T}\}$. To see this, it is enough to note that 0 is a limit point of B, for if V is an open set such that $0 \in V$ then $(-1, 1) \subset V$ so that $V \cap B = B \neq \varnothing$.

(iii) The space $\{A, \mathscr{T}\}$ is not a Hausdorff space. By S Proposition 5.4.2, each compact subspace of a Hausdorff space is closed. In part (ii), a compact subspace of $\{A, \mathscr{T}\}$ was exhibited which is not closed. Thus $\{A, \mathscr{T}\}$ is not a Hausdorff space.

6.3 Consider the collection $\mathscr{V}$ of subsets of A, which consists of $[0, \frac{1}{2})$ and the singleton subsets $\{x\}$ for $\frac{1}{2} \leq x \leq 1$. Then $\mathscr{V}$ is a disjoint collection with union equal to A. Moreover, each member of $\mathscr{V}$ is open in $\{A, \mathscr{T}\}$. Thus $\mathscr{V}$ is an open cover of $\{A, \mathscr{T}\}$. But if any member of $\mathscr{V}$ (which contains infinitely many open sets) is omitted, then the resulting collection of sets is not a cover of A, and so $\mathscr{V}$ contains no finite subcover. Therefore $\{A, \mathscr{T}\}$ is not a compact space.

6.4 (i) Since $\mathscr{B}$ is a basis for the topology $\mathscr{T}$, it follows that each open set of $\{A, \mathscr{T}\}$ is a union of members of $\mathscr{B}$. Thus if $V \in \mathscr{T}$ and $2 \in V$, then there exists B in $\mathscr{B}$ such that $2 \in B \subset V$. But the only members of $\mathscr{B}$ which contain 2 are B_2 and A (and, of course, $B_2 \subset A$). Thus if $V \in \mathscr{T}$ and $2 \in V$, then $B_2 \subset V$.

(ii) Let $\mathscr{V}$ be an open cover of $\{A, \mathscr{T}\}$. We must show that $\mathscr{V}$ contains a finite subcover. First we note that there exists $V \in \mathscr{V}$ such that $2 \in V$. By part (i), $B_2 \subset V$. Also, there exists V' in $\mathscr{V}$ such that $3 \in V'$. Since B_3 and A are the only members of $\mathscr{B}$ which contain 3, it follows from an argument similar to that of part (i) that $B_3 \subset V'$. Since $B_2 \cup B_3 = A$, we see that $V \cup V' = A$. Thus $\{V, V'\}$ is a finite subcover of $\mathscr{V}$. Since each open cover of $\{A, \mathscr{T}\}$ contains a finite subcover, the space $\{A, \mathscr{T}\}$ is compact.

6.5 (i) The only open set of $\{\mathbf{N}, \mathscr{T}\}$ which contains 1 is $\mathbf{N}$. Thus $\mathbf{N}$ must be a member of each open cover of $\{\mathbf{N}, \mathscr{T}\}$, and so each open cover of $\{\mathbf{N}, \mathscr{T}\}$ contains a finite subcover, namely the subcover whose only member is $\mathbf{N}$.

(ii) Since

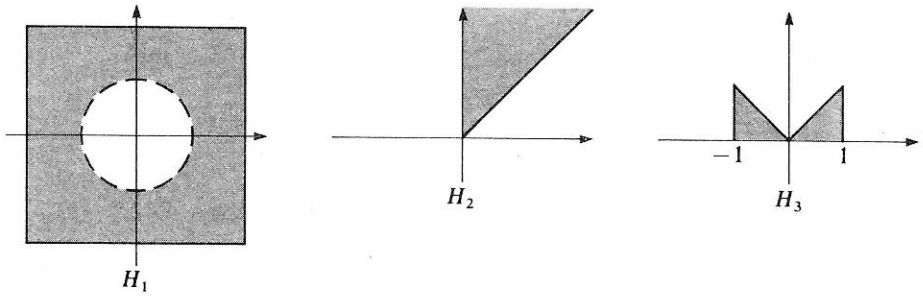
$$\mathbf{O} = \mathbf{N} - B_2,$$

it follows that $\mathbf{O}$ is closed in $\{\mathbf{N}, \mathscr{T}\}$. By S Proposition 5.6.1, each closed subspace of a compact space is compact. Thus $\mathbf{O}$ is a compact subset of $\{\mathbf{N}, \mathscr{T}\}$. (Alternatively, a similar argument to part (i) shows that each open cover of $\mathbf{O}$ includes the set $\mathbf{O}$ itself, which is open in the subspace $\mathbf{O}$, which thus forms a finite subcover.)

(iii) The argument to show that the subspace $\mathbf{E}$ is compact is similar to that of part (i); the only open set of the subspace $\mathbf{E}$ which contains 2 is $\mathbf{E}$ itself. The set $\mathbf{E}$ is not closed in $\{\mathbf{N}, \mathscr{T}\}$, for every member of the basis of $\mathscr{T}$ has a non-empty intersection with $\mathbf{E}$, and so every odd positive integer is a limit point of $\mathbf{E}$.

(iv) The space $\{\mathbf{N}, \mathscr{T}\}$ is not Hausdorff, since $\mathbf{E}$ is a compact subset of $\{\mathbf{N}, \mathscr{T}\}$ which is not closed.

6.6



The set H_1 is (a) not closed, (b) bounded, (c) not compact. H_2 is (a) closed, (b) not bounded, (c) not compact. H_3 is (a) closed, (b) bounded, (c) compact.

In each case, the justification for the answer to (c) is that a subset of Euclidean space $\mathbf{R}^2$ is compact if and only if it is closed and bounded.

6.7 (i) First we note that $e(0, x) = |x|$ for all $x \in \mathbf{R}$. Thus $B_r(0) = (-r, r)$ for each positive real number r. Also, $e(a, x) \geq |a|$ for all distinct real numbers a and x, so that if $a \neq 0$, then $B_{|a|}(a) = \{a\}$.

Let V be a subset of $\mathbf{R}$. If V is open in M and $0 \in V$, then there exists $r > 0$ such that $B_r(0) \subset V$. Thus $(-r, r) \subset V$ for some $r > 0$. Therefore the stated condition is necessary for V to be open in M.

It is also sufficient. For if $0 \notin V$, then V is open in M, since $B_{|a|}(a) \subset V$ for each a in V; whilst if $(-r, r) \subset V$ for some $r > 0$, then V is open in M, since $B_r(0) \subset V$ and $B_{|a|}(a) \subset V$ for each a in V such that $a \neq 0$.

Thus a subset V of $\mathbf{R}$ is open in M if and only if either $0 \notin V$ or $(-r, r) \subset V$ for some positive real number r.

(ii) Let H be a subset of $\mathbf{R}$ such that $0 \notin H$. The set H is closed in M if and only if $\mathbf{R} - H$ is open in M. Since $0 \notin H$, we see that $0 \in \mathbf{R} - H$ and so, by part (i), $\mathbf{R} - H$ is open in M if and only if $(-r, r)$ is a subset of $\mathbf{R} - H$ for some $r > 0$. This is the case if and only if $x \notin (-r, r)$ for all $x \in H$, or equivalently if and only if $|x| \geq r$ for all $x \in H$.

(iii) Put $F = [1, 2]$. Since $x \geq 1$ for all $x \in F$, it follows from part (ii) that F is closed in M. Since $e(0, x) = |x|$ for $x \in \mathbf{R}$, a subset of $\mathbf{R}$ is bounded in the metric space M if and only if it is bounded in the usual sense. Thus F is a closed bounded subset of the metric space M.

But the subspace F of M is not compact. For each a in F, the singleton set $\{a\}$ is open in M, since $\{a\} = B_{|a|}(a)$, and hence is open in F. Thus $(\{a\})_{a \in F}$ is an open cover of the subspace F which evidently contains no finite subcover.

(iv) Put

$$K = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}.$$

By S Definition 5.2.4, page 79, to prove that K is compact in M it is enough to show that if $\mathscr{V}$ is a collection of open sets of M whose union contains K, then the union of finitely many members of $\mathscr{V}$ contains K. Let $\mathscr{V}$ be a collection of open sets of M with union containing K. Then there exists $V_0 \in \mathscr{V}$ such that $0 \in V_0$. By part (i),

there exists $r > 0$ such that $(-r, r) \subset V_0$. Also, there exists a positive integer m such that $\frac{1}{m} \leq r$. Thus, for all $n > m$, we have $\frac{1}{n} < \frac{1}{m} \leq r$ and hence $\frac{1}{n} \in (-r, r) \subset V_0$. Thus V_0 contains all elements of K except possibly $1, \frac{1}{2}, \ldots, \frac{1}{m}$. For each $j = 1, \ldots, m$ there exists $V_j \in \mathscr{V}$ such that $\frac{1}{j} \in V_j$. Then

$$K \subset V_0 \cup V_1 \cup \ldots \cup V_m.$$

Hence K is compact in M.

7.1 (i) Since the disjoint open sets $\{a\}$ and $\{b, c, d, e\}$ have union equal to A, they form a partition of the topological space $\{A, \mathscr{T}\}$, and it follows from **S** Proposition 6.2.3, page 94, that $\{A, \mathscr{T}\}$ is not connected.

(ii) Put $H = \{b, d, e\}$. The induced topology on H is

$$\{\varnothing, \{d\}, H\}.$$

Thus the set of closed sets of the subspace H is

$$\{H, \{b, e\}, \varnothing\}.$$

Since the only subsets of the subspace H which are both open and closed are $\varnothing$ and H, it follows from **S** Corollary 6.2.4, page 95, that the subspace H is connected.

(iii) Since $H = \{b, d, e\}$ is connected, it follows from **S** Proposition 6.2.18, page 99, that $Cl(H)$ is connected.

The closed sets of $\{A, \mathscr{T}\}$ are

$$A, \{b, c, d, e\}, \{a, b, e\}, \{b, e\}, \{a\}, \varnothing.$$

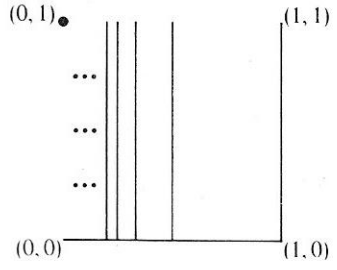
Hence

$$Cl(H) = \{b, c, d, e\}.$$

Since $\{A, \mathscr{T}\}$ is not a connected space, we see that $\{b, c, d, e\}$ is a maximal connected subspace of $\{A, \mathscr{T}\}$, and so is a component.

Thus the components of $\{A, \mathscr{T}\}$ are $\{a\}$ and $\{b, c, d, e\}$.

7.2 A sketch of the subspace H should be made.



It follows from Proposition 3, *Unit* 7, page 65, that a line segment joining two points of $\mathbf{R}^2$ is connected. Thus I is connected and each K_n is connected. Since $I \cap K_n \neq \varnothing$ for each positive integer n, the subspace

$$X = I \cup \bigcup_{n=1}^{\infty} K_n$$

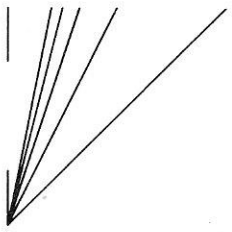
of $\mathbf{R}^2$ with the usual topology is connected, by **S** Corollary 6.2.16, page 98.

Put $a = (0, 1)$. We show next that a is a limit point of X. If $r > 0$, then there exists a positive integer m such that $\frac{1}{m} < r$. Then $\left(\frac{1}{m}, 1\right) \in B_r(a)$, and so $B_r(a) \cap X \neq \varnothing$ for each $r > 0$. Thus

$$X \subset H \subset Cl(X).$$

It now follows from **S** Proposition 6.2.18, page 99, that H is a connected subspace of $\mathbf{R}^2$.

7.3 Again we begin with a sketch of the subset K.



For each positive integer n, the line segment L_n is connected. As $(0,0) \in L_n$ for each n, then by **S** Proposition 6.2.15, page 98, the subspace

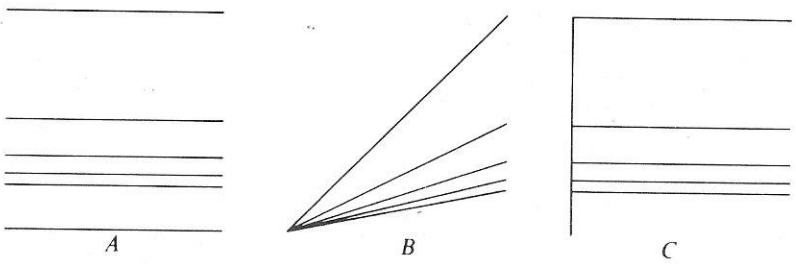
$$X = \bigcup_{n=1}^{\infty} L_n$$

of $\mathbf{R}^2$ is connected. Since each point of J is a limit point of X, it follows that

$$X \subset K \subset Cl(X),$$

and again K is a connected subspace of $\mathbf{R}^2$ by **S** Proposition 6.2.18.

7.4 We are concerned with the following subspaces of $\mathbf{R}^2$.



First we note that the subspaces B and C are connected, whilst A is not connected since each of the line segments K_n is open and closed in A. Thus, since connectedness is a topological property (**S** Corollary 6.2.11, page 97), A is not homeomorphic to either B or C.

It remains to show that the subspaces B and C are not homeomorphic. Suppose, to obtain a contradiction, that there is a homeomorphism $f: B \longrightarrow C$. If $f(0,0) = c \in C$, then the subspaces $B - \{(0,0)\}$ and $C - \{c\}$ are homeomorphic. But $B - \{(0,0)\}$ has infinitely many components, whilst for all c in C, $C - \{c\}$ has at most three components. Since (**S**, page 104, line -6) the number of components of a topological space is a topological invariant, we have a contradiction. Thus the subspaces B and C are not homeomorphic.

The last part of the above solution is an example of the general technique described in Note (viii), *Unit 7*, page 72.

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